1.

(1) Find the limits. If limit does not exist, prove it.

\[ \lim_{x \to 0} \frac{2}{2 + 2^x} \]

\[ \lim_{x \to 0} (1 + 2x + 3x^2)^{\frac{1}{x}} \]

(2) Find the extrema of the following function.

\[ z = f(x, y) = x^3 - 9xy + y^3 \]
2.

(1) The function $G$ is defined as follow:
$G(x, y, z) = 3x^2 + 2yz + xz$
(a) What are the gradient, the Hessian and the Laplacian of $G$?
(b) Find the directional derivative in the direction of unit vector $u = (0, 0, 1)$ at the point $(1, 1, 0)$. Interpret this result.

(2) Consider the surface given by:
$G(x, y, z) = 3x^2 + 2yz + xz = C$ (C is a constant)
(a) Find the unit normal vector at the point $(1, 1, 0)$.
(b) Find the equation of the tangent plan at the point $(1, 1, 0)$.
3.

(1) Answer the following questions for the differential equation:

\[ x'(t) + t \cdot x(t) = t \cdot x(t)^3 \]

(a) When \( z(t) = x(t)^2 \), find the differential equation for \( z(t) \).

(b) Solve the differential equation in (a).

(c) Solve the differential equation for \( x(t) \).

(2) When \( x > 0 \), solve the following differential equation:

\[ y''(x) + \frac{4}{x} \cdot y'(x) + \frac{2}{x^2} \cdot y(x) = 0 \]
4. Answer the following questions for the matrix $A$.

$$A = \begin{bmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{bmatrix}$$

(1) Find all the eigenvalues and eigenvectors of which the norms are unity.

(2) Find the coefficients $a_3, a_2, a_1$ and $a_0$ of characteristic polynomial

$$\det(A - \lambda E) = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0,$$

where $E$ is the unit matrix.

(3) Calculate the following equation with the coefficients obtained in the problem (2).

$$a_3A^3 + a_2A^2 + a_1A + a_0E$$

(4) Compute the inverse matrix $A^{-1}$ by using the result of problem (3).
5. The probability density function of the random variable \( X \) is denoted by

\[
f(x) = ae^{-\lambda|x|},
\]
where \( a \) and \( \lambda \) are positive constants. Answer the following questions. Use the following formula if necessary.

\[
\int_0^\infty x^2 e^{-\lambda x} dx = \frac{2}{\lambda^3} \quad (\lambda > 0).
\]

(1) Find the values of \( a \) and \( \lambda \) such that the variance of \( X \) is 1.
(2) Find the cumulative distribution function \( F(x) \) of \( X \) using the values of \( a \) and \( \lambda \) found in the problem (1).
(3) Find the cumulative distribution function of \( Y = X^2 \).